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AN ACCOUNT OF CAUCHY'S "CALCUL DES RESIDUS."

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§ 1.

THIS very curious method for evaluating a great variety of definite integrals or for establishing relations between more complex and simple ones of the same class, is founded on the theorem that, if in a double integral the limits of both integrations are independent, the same result is obtained if the order of integration is inverted, provided the element function does not become infinite for one or more systems of simultaneous values of the variables comprised within the limits of integration.

Thus

$$\int_{x_0}^{x_n} dx \int_{y_0}^{y_n} dy\, f(x, y) = \int_{y_0}^{y_n} dy \int_{x_0}^{x_n} dx\, f(x, y), \tag{1}$$

is an identical equation if $f(x, y)$ does not become infinite for certain values x_1 and y_1 for which

$$\left.\begin{array}{ll} & x_n > x_1 > x_0\,;\ y_n > y_1 > y_0, \\ \text{or} & x_n > x_1 > x_0\,;\ y_n > y_1 = y_0, \\ \text{or} & x_n > x_1 > x_0\,;\ y_n = y_1 > y_0, \\ \text{or} & x_n > x_1 = x_0\,;\ y_n > y_1 > y_0, \\ \text{or} & x_n = x_1 > x_0\,;\ y_n > y_1 > y_0. \end{array}\right\} \tag{2}$$

But suppose $f(x_1, y_1) = \infty$ then the corresponding element $dxdyf(x_1, y_1)$ is of the form $0 \times \infty$ and may therefore be zero, a finite quantity or infinity. In the first case only, the integration by the regular process is correct, but in the second and third case it is not because the element, although finite or infinite, is treated as an infinitesimal and therefore neglected in comparison with the sum of the remaining elements. To establish identity in equation (1) in that case, a correction must be added. This is called by Cauchy the correction for discontinuity and the process to determine it the "Calcul des Residus."

To illustrate, let us take the double integral

$$u=\int_{-b}^{a}dx\int_{-d}^{c}dy\frac{y^2-x^2}{(x^2+y^2)^2}=-\tan^{-1}\frac{a}{c}-\tan^{-1}\frac{b}{c}-\tan^{-1}\frac{a}{d}-\tan^{-1}\frac{b}{d},$$

$$u'=\int_{-d}^{c}dy\int_{-b}^{a}dx\frac{y^2-x^2}{(x^2+y^2)^2}=\tan^{-1}\frac{c}{a}+\tan^{-1}\frac{c}{b}+\tan^{-1}\frac{d}{a}+\tan^{-1}\frac{d}{b};$$

therefore $u'-u=4\times\frac{1}{2}\pi=2\pi$. The cause of this is the discontinuous element for $x=y=0$, and we have 2π for the correction for discontinuity, which is here determined by common integration.

However, in the application of these principles to the evaluation of definite integrals these integrations cannot in general be performed and it will be shown that the corrections for discontinuity may, nevertheless, be determined in many cases.

§ 2.

Let z be some simple function of x and yi, where $i=\sqrt{-1}$, such as $x+yi$; $x(a+yi)$; xe^{yi}; xe^{a+yi} and others, then if $f(z)$ is such a function that

$$\frac{d}{dy}\left\{\frac{dz}{dx}f(z)\right\}=\frac{d}{dx}\left\{\frac{dz}{dy}f(z)\right\}$$

we have, if there is no discontinuous element, the identity

$$\int_{x_0}^{x_n}dx\int_{y_0}^{y_n}dy\frac{d}{dy}\left\{\frac{dz}{dx}f(z)\right\}=\int_{y_0}^{y_n}dy\int_{x_0}^{x_n}dx\frac{d}{dx}\left\{\frac{dz}{dy}f(z)\right\}, \tag{3}$$

or

$$\int_{x_0}^{x_n}dx\left[\frac{dz}{dx}f(z)\right]_{y_0}^{y_n}=\int_{y_0}^{y_n}dy\left[\frac{dz}{dy}f(z)\right]_{x_0}^{x_n}. \tag{3'}$$

But if $f(z_1)=\infty$ and x_1 and y_1 are the corresponding values of x and y, then if Δx denotes an increment to x_1 we have the following perfect identities for the first case in (2), viz., $x_n>x_1>x_0$; $y_n>y_1>y_0$.

$$\int_{x_1+\Delta x}^{x_n}dx\left[\frac{dz}{dx}f(z)\right]_{y_0}^{y_n}=\int_{y_0}^{y_n}dy\left[\frac{dz}{dy}f(z)\right]_{x_1+\Delta x}^{x_n},$$

$$\int_{x_0}^{x_1-\Delta x}dx\left[\frac{dz}{dx}f(z)\right]_{y_0}^{y_n}=\int_{y_0}^{y_n}dy\left[\frac{dz}{dy}f(z)\right]_{x_0}^{x_1-\Delta x}.$$

Adding these we have

$$\int_{x_0}^{x_n}dx\left[\frac{dz}{dx}f(z)\right]_{y_0}^{y_n}-\int_{x_1-\Delta x}^{x_1+\Delta x}dx\left[\frac{dz}{dx}f(z)\right]_{y_0}^{y_n}$$

$$=\int_{y_0}^{y_n}dy\left[\frac{dz}{dy}f(z)\right]_{x_0}^{x_n}-\int_{y_0}^{y_n}dy\left[\frac{dz}{dy}f(z)\right]_{x_1-\Delta x}^{x_1+\Delta x} \tag{4}$$

At the limit when $\Delta x = 0$ we have

$$\int_{x_1-0}^{x_1+0} dx\left[\frac{dz}{dx}f(z)\right]_{y_0}^{y_n} = 0.$$

The term

$$\int_{y_0}^{y_n} py\left[\frac{dz}{dy}f(z)\right]_{x_1-0}^{x_1+0}$$

would likewise vanish were it not for the discontinuous element

$$dy\left[\frac{dz}{dy}f(z)\right]_{z=z_1}$$

This term is then the correction for discontinuity, and if z_1 is the only value that makes $f(z) = \infty$ we have the identical equation:

$$\int_{x_0}^{x_n} dx\left[\frac{dz}{dx}f(z\right]_{y_0}^{y_n} = \int_{y_0}^{y_n} dy\left[\frac{dz}{dy}f(z)\right]_{x_0}^{x_n} - \int_{y_0}^{y_n} dy\left[\frac{dz}{dy}f(z)\right]_{x_1-\Delta x}^{x_1+\Delta x}. \qquad (\Delta x = 0) \quad (4')$$

If there are other discontinuous elements for $z_2, z_3, \ldots z_r$ we must correct for each and add the corrections.

§ 3.

Let us assume the simplest form for z, viz.,

$$z = x+yi. \qquad (5)$$

With this we have in (4′), since $\frac{dz}{dx} = 1$ and $\frac{dz}{dy} = i$,

$$\int_{x_0}^{x_n} dx[f(x+y_n i)-f(x+y_0 i)] = i\int_{y_0}^{n} dy[f(x_n+yi)-f(x_0+yi)]$$

$$- i\int_{y_0}^{y_n} dy[f(x_1+\Delta x+yi)-f(x_1-\Delta x+yi)]. \qquad (6)$$

Let the equation

$$[f(z)]^{-1} = 0 \qquad (7)$$

have m roots $= z_1$* so that we may place

$$f(z) = \frac{\phi(z)}{(z-z_1)^m}. \qquad (8)$$

Let us denote the correction of discontinuity for the m-tuple root $z_1 = x_1 + y_1 i$ by $\Delta_{(x_1+y_1 i)^m}$ then we have, using the form (8) in (6),

$$\Delta_{(x_1+y_1 i)^m} = i\int_{y_0}^{y_n} dy\left[\frac{\phi(x_1+\Delta x+yi)}{[\Delta x+(y-y_1)i]^m} - \frac{\phi(x_1-\Delta x+yi)}{[-\Delta x+(y-y_1)i]^m}\right]$$

$$= i\int_{y_0}^{y_n} dy\left[\frac{\phi[x_1+y_1 i+\Delta x+(y-y_1)i]}{[\Delta x+(y-y_1)i]^m} - \frac{\phi[x_1+y_1 i-\Delta x+(y-y_1)i]}{[-\Delta x+(y-y_1)i]^m}\right].$$

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*This is not the only supposition that can be made but so far it is the only case to which Cauchy's method applies.

Developing the numerator of both terms by Taylor's series we have

$$\Delta_{(x_1+y_1i)^m} = i\int_{y_0}^{y_n} dy \sum_{k=1}^{k=\infty}\left[\frac{\phi^{(k-1)}(x_1+y_1i)}{(k-1)!}\left\{[\Delta x+(y-y_1)i]^{-(m-k+1)} - [-\Delta x+(y-y_1)]^{-(m-k+1)}\right\}\right]$$

and integrating this we obtain

$$\Delta_{(x_1+y_1i)^m} = \sum_{k=1}^{k=\infty}\left[\frac{\phi^{(k-1)}(x_1+y_1i)}{(k-1)!}\cdot\frac{[\Delta x+(y-y_1)i]^{k-m}-[-\Delta x+(y-y_1)i]^{k-m}}{k-m}\right]_{y_0}^{y_n}. \qquad (9)$$

If we put $\Delta x = 0$ both terms of the general term of this series are perfectly identical and destroy each other as long as k is not equal to m when it becomes indeterminate. The series reduces then to the mth term taken between the limits y_n and y_0 or

$$\Delta_{(x_1+y_1i)^m} = \frac{\phi^{(m-1)}(x_1+y_1i)}{(m-1)!}\left[\left\{\frac{[\Delta x+(y-y_1)i]^{k-m}-[-\Delta x+(y-y_1)i]^{k-m}}{k-m}\right\}_{k=m}\right]_{y_0}^{y_n}$$

$$= \frac{\phi^{(m-1)}(x_1+y_1i)}{(m-1)!}\left[\log\frac{\Delta x+(y-y_1)i}{-\Delta x+(y-y_1)i}\right]_{y_0}^{y_n}.$$

Place $\tan\varphi = \frac{y-y_1}{\Delta x}$ then $\tan(\pi-\varphi) = \frac{y-y_1}{-\Delta x}$ and

$$\Delta x+(y-y_1)i = \frac{\Delta x}{\cos\varphi}e^{\phi i},$$

$$-\Delta x+(y-y_1)i = \frac{\Delta x}{\cos\varphi}e^{(\pi-\phi)i},$$

$$\varphi_n = \tan^{-1}\frac{y_n-y_1}{\Delta x} = \tan^{-1}(\infty) = \frac{\pi}{2},$$

$$\varphi_0 = \tan^{-1}\frac{y_0-y_1}{\Delta x} = \tan^{-1}(-\infty) = -\frac{\pi}{2};$$

therefore

$$\Delta_{(x_1+y_1i)^m} = \frac{\phi^{(m-1)}(x_1+y_1i)}{(m-1)!}\left[\log\frac{e^{\phi i}}{e^{(\pi-\phi)i}}\right]_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}$$

$$= \frac{\phi^{(m-1)}(x_1+y_1i)}{(m-1)!}\left[(2\varphi-\pi)i\right]_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}$$

$$= 2\pi i\frac{\phi^{(m-1)}(x_1+y_1i)}{(m-1)!}. \qquad (10)$$

If however $y_0 = y_1$ the result is quite different. We have then, replacing y_1 by y_0 in (9),

$$\varDelta_{(x_1+y_0i)^m}=\sum_{k=1}^{k=\infty}\left[\frac{\psi^{(k-1)}(x_1+y_0i)}{(k-1)!}\cdot\frac{[\varDelta x+(y-y_1)i]^{k-m}-[-\varDelta x+(y-y_1)i]^{k-m}}{k-m}\right]_{y_0}^{y_n}.$$

Here as before all terms of this series vanish for the upper limit except the mth. For the lower limit all terms vanish for which $k-m$ is an even number, zero excepted, and the terms for which $k-m$ is odd and $k<m$ are infinite if $\varDelta x=0$. The mth term is, as we shall see, finite; it must be retained because it is not necessarily homogeneous with the infinite terms.

Let $m=2n$ then

$$\varDelta_{(x_1+y_0i)^{2n}}=\frac{2\psi(x_1+y_0i)}{2n-1}\varDelta x^{-2n+1}+\frac{\psi^{(2n-1}(x_1+y_0i)}{(2n-1)!}$$
$$\times\left[\log\frac{\varDelta x+(y-y_0)i}{-\varDelta x+(y-y_0)i}\right]_{y_0}^{y_n}$$
$$=\frac{2\psi(x_1+y_0i)}{2n-1}\varDelta x^{-2n+1}+\frac{\psi^{(2n-1)}(x_1+y_0i)}{(2n-1)!}\Big[(2\varphi-\pi)i\Big]_0^{\frac{\pi}{2}},\ \text{if}\ \tan\varphi=\frac{y-y_0}{\varDelta x}$$
$$=\frac{2\psi(x_1+y_1i)}{2n-1}\infty^{2n-1}+\pi i\frac{\psi^{(2n-1)}(x_1+y_0i)}{(2n-1)!}. \qquad (10'_1)$$

Also if $m=2n+1$ we have, since the first term vanishes for the lower limit,

$$\varDelta_{(x_1+y_0i)^{2n+1}}=\frac{2\psi'(x_1+y_0i)}{2n-1}\varDelta x^{-2n+1}+\frac{\psi^{(2n)}(x_1+y_0i)}{2n!}$$
$$\times\left[\log\frac{\varDelta x+(y-y_0)i}{-\varDelta x+(y-y_0)i}\right]_{y_0}^{y_n}$$
$$=\frac{2\psi'(x_1+y_0i)}{2n-1}\infty^{2n-1}+\pi i\frac{\psi^{(2n)}(x_1+y_0i)}{2n!}. \qquad (10'_2)$$

If $n=0$ the correction is

$$\varDelta_{(x_1+y_0i)}=\pi i\psi(x_1+y_0i). \qquad (10'_3)$$

The form which is given to the first term of $(10'_1)$ and $(10'_2)$ I have preferred to simply writing ∞, since it shows at least for $(10'_2)$ that this term may yet be zero, viz., if $\psi'(x_1+y_0i)=0$, which is the case if $\psi(z)=\text{const.}$ This term is however omitted by Bierens de Haan in his large work on definite integrals: Exposee de la theorie des propriete'es, des formules de transformation et des methodes d' evaluation des integrales definies, pages 45, 46 and page 666. At the proper place I shall give a test case which proves the necessity of this term.

In a similar manner the corrections for discontinuity for the remaining cases of (2) are found. I shall set them down without analysis:

$$\varDelta_{(x_1+y_n i)^{2n}} = -\frac{2\psi(x_1+y_n i)}{2n-1}\infty^{2n-1} + \pi i \frac{\psi^{(2n-1)}(x_1+y_n i)}{(2n-1)!} \qquad (10''_1)$$

$$\varDelta_{(x_1+y_n i)^{2n+1}} = -\frac{2\psi'(x_1+y_n i)}{2n-1}\infty^{2n-1} + \pi i \frac{\psi^{(2n)}\quad(x_1+y_1 i)}{2n!} \qquad (10''_2)$$

$$\varDelta_{(x_1+y_n i)} = \pi i \psi(x_1+y_n i). \qquad (10''_3)$$

To determine the correction if $x_1 = x_0$ or $x_1 = x_n$ we have to divide the limits of the y-integration into two parts viz., $\Big]_{y_1+\triangle y}^{y_n}$ and $\Big]_{y_0}^{y_1-\triangle y}$

We shall then find

$$\varDelta_{(x_0+y_1 i)^{2n}} = \frac{2i\psi(x_0+y_1 i)}{2n-1}\infty^{2n-1} + \pi i \frac{\psi^{(2n-1)}(x_0+y_1 i)}{(2n-1)!}, \qquad (10'''_1)$$

$$\varDelta_{(x_0+y_1 i)^{2n+1}} = \frac{2i\psi'(x_0+y_1 i)}{2n-1}\infty^{2n-1} + \pi i \frac{\psi^{(2n)}\quad(x_0+y_1 i)}{2n!}, \qquad (10'''_2)$$

$$\varDelta_{x_0+y_1 i} = \pi i \psi(x_0+y_1 i), \qquad (10'''_3)$$

$$\varDelta_{(x_n+y_1 i)^{2n}} = -\frac{2i\psi(x_n+y_1 i)}{2n-1}\infty^{2n-1} + \pi i \frac{\psi^{(2n-1)}(x_n+y_1 i)}{(2n-1)!}, \qquad (10''''_1)$$

$$\varDelta_{(x_n+y_1 i)^{2n+1}} = -\frac{2i\psi(x_n+y_1 i)}{2n-1}\infty^{2n-1} + \pi i \frac{\psi^{(2n)}\quad(x_n+y_1 i)}{2n!}, \qquad (10''''_2)$$

$$\varDelta_{x_n+y_1 i} = \pi i \psi(x_n+y_1 i). \qquad (10''''_3)$$

If $x_1 = x_0$ and at the same time $y_1 = y_0$ or $y_1 = y_n$; again if $x_1 = x_n$ and at the same time $y_1 = y_0$ or $y_1 = y_n$ equation (3′) is correct because in both members the discontinuous elements are included and not lost. These integrals, though they may be finite or infinite, are identical and need no correction.* We have then

$$\varDelta_{(x_0+y_0 i)^m} = \varDelta_{(x_0+y_n i)^m} = \varDelta_{(x_n+y_0 i)^m} = \varDelta_{(x_n+y_n i)^m} = 0. \qquad (10^{\text{v}})$$

§4.

We are now prepared to make some applications. There are ten special assumptions made originally by Cauchy for the limits x_n, x_0, y_n, y_0 (Bierens de Haan, pages 667, 668). Of these the most useful (so considered by Cauchy himself) is that for which $x_n = \infty$; $x_0 = -\infty$; $y_n = \infty$; $y_0 = 0$. We have then in (6), denoting the sum of all the necessary corrections, as the formulas (10) give them for each special case, simply, by $\varDelta$

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*Bierens de Haan gives in this case $\pm \infty$ for the correction of discontinuity. Price in his Infinitesimal Calculus Vol. II, does not explicitly treat of this case. (See p. 129 at bottom.)

$$\int_{-\infty}^{\infty} dx[f(x+\infty i)-f(x+0.i)]=i\int_{0}^{\infty} dy[f(\infty+yi)-f(-\infty+yi)]-\Delta.$$

If now $f(z)$ is such that

$$f(x+\infty i)=f(\infty+yi)=f(-\infty+yi)=0,^{*} \tag{11}$$

then we have

$$\int_{-\infty}^{\infty} dxf(x)=\Delta. \tag{12}$$

We have

$$\int_{-\infty}^{\infty} dxf(x)=\int_{0}^{\infty} dxf(x)+\int_{-\infty}^{0} dx\,f(x)$$

$$=\int_{0}^{\infty} dx[f(x)+f(-x)]=\Delta. \tag{13}$$

Also

$$\int_{0}^{\infty} dx[f(x)+f(-x)]=\int_{1}^{\infty} dx[f(x)+f(-x)]+\int_{0}^{1} dx[f(x)+f(-x)]$$

$$=\int_{0}^{1} dx\left[f(x)+f(-x)+\frac{1}{x^2}\left\{f\left(\frac{1}{x}\right)+f\left(-\frac{1}{x}\right)\right\}\right]$$

$$=\Delta. \tag{14}$$

Or place in (13) $x=\dfrac{1-x'}{x'}$ then we have, omitting primes,

$$\int_{0}^{1}\frac{dx}{x^2}\left[f\left(\frac{1-x}{x}\right)+f\left(\frac{x-1}{x}\right)\right]=\Delta. \tag{15}$$

Or place in (12) and (13) $x=\tan\varphi$, then

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\varphi}{\cos^2\varphi}f(\tan\varphi)=\Delta. \tag{16}$$

$$\int_{0}^{\frac{\pi}{2}}\frac{d\varphi}{\cos^2\varphi}[f(\tan\varphi)+f(-\tan\varphi)]=\Delta. \tag{17}$$

These and a variety of other forms have the identical correction for discontinuity.

Ex. 1. Required $\displaystyle\int_{-\infty}^{\infty}\frac{\varphi(x)}{(a+x)^m}dx.$

We have here $f(z)=\dfrac{\varphi(x+yi)}{(a+x+yi)^m}$, and if $\dfrac{\varphi(\pm\infty+yi)}{(a\pm\infty+yi)^m}=\dfrac{\varphi(x+\infty i)}{(a+x+\infty i)^m}=0$ condition (11) is satisfied. If $\varphi(z)$ does not become infinite for any value $z_1=x_1+y_1i$ within the limits of the x- and y-integrations we have an m-tuple discontinuity for $z_1=-a+0.i$, since it makes

$$\left.\frac{\varphi(z)}{(a+z)^m}\right]_{z=-a}=\infty.$$

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[*] These conditions can hardly be called restrictions; they are nothing else than essential though not sufficient conditions for the convergence of the series which the integral represents. At least without these no finite value of the integral is possible.

We have then $x_1 = -a$ and $y_1 = 0$, therefore, since

$$\infty > -a > -\infty;\ \infty > 0 = 0,$$

we have to use formulæ $(10'_1)$, $(10'_2)$, $(10'_3)$ according as $m = 2n$, $2n+1$ or $= 1$. We have then

$$\int_{-\infty}^{\infty}\frac{\varphi(x)}{(a+x)^{2n}}dx = \frac{2\varphi(-a)}{2n-1}\infty^{2n-1}+\pi i\frac{\varphi^{(2n-1)}(-a)}{(2n-1)!}, \qquad (a_1)$$

$$\int_{-\infty}^{\infty}\frac{\varphi(x)}{(a+x)^{2n+1}}dx = \frac{2\varphi'(-a)}{2n-1}\infty^{2n-1}+\pi i\frac{\varphi^{(2n)}(-a)}{2n!}, \qquad (a_2)$$

$$\int_{-\infty}^{\infty}\frac{\varphi(x)}{a+x}dx = \pi i\varphi(-a). \qquad (a_3)$$

1. Let $\varphi(z) = e^{mzi}$. This becomes $= \infty$ for $z = 0 - \infty i$. Now although the value 0 for x is included within the x- integration, the value $-\infty$ for y is not; there is consequently no correction for this value of z. Now since

$$\frac{e^{b(+or-\infty+yi)i}}{(a\pm\infty+yi)^m} = \frac{e^{-by}[\cos\{b(\pm\infty)\}+i\sin\{b(\pm\infty)\}]}{(a+\infty+yi)^m} = 0,$$

$$\frac{e^{b(x+\infty i)i}}{(a+x+\infty i)^m} = \frac{e^{-b\infty}(\cos bx + i\sin bx)}{(a+x+\infty i)^m} = 0$$

condition (11) is satisfied and we can safely use formulæ (a_1), (a_2) and (a_3). We have then, since $\varphi^{(m-1)}(-a) = (bi)^{m-1}e^{-abi}$,

$$\int_{-\infty}^{\infty}\frac{e^{bxi}\,dx}{(a+x)^{2n}} = \frac{2e^{-abi}}{2n-1}\infty^{2n-1}+\pi i\frac{(bi)^{m-1}}{(2n-1)!}e^{-abi}$$

$$= \frac{2e^{-abi}}{2n-1}\infty^{2n-1}. \qquad (b_1)$$

(Omitting here the second term being finite and homogeneous with the first, which is infinite.)

$$\int_{-\infty}^{\infty}\frac{e^{bxi}\,dx}{(a+x)^{2n+1}} = \frac{2bi\,e^{-abi}}{2n-1}\infty^{2n-1}. \qquad (b_2)$$

$$\int_{-\infty}^{\infty}\frac{e^{bxi}\,dx}{a+x} = \pi i\,e^{-abi}. \qquad (b_3)$$

Separating the real from the imaginary in (b_3) we have

$$\int_{-\infty}^{\infty}\frac{\cos bx}{a+x}dx = \pi\sin ab, \qquad (b'_3)$$

$$\int_{-\infty}^{\infty}\frac{\sin bx}{a+x}dx = \pi\cos ab. \qquad (b''_3)$$

Placing $b = 0$ in (b_1), (b_2), (b'_3) and $a = 0$ in (b''_3) we have

$$\int_{-\infty}^{\infty}\frac{dx}{(a+x)^{2n}} = \frac{2}{2n-1}\infty^{2n-1} = \infty, \qquad (b'''_1)$$

$$\int_{-\infty}^{\infty} \frac{dx}{(a+x)^{2n-1}} = 0, \qquad (b_2''')$$

$$\int_{-\infty}^{\infty} \frac{dx}{a+x} = 0, \qquad (b_3''')$$

$$\int_{-\infty}^{\infty} \frac{\sin bx}{x}\,dx = 2\int_{0}^{\infty} \frac{\sin bx}{x}\,dx = \pi. \qquad (b_4''')$$

We can verify (b_1'''), (b_2''') and (b_3''') by common integration. We have

$$\int_q^p \frac{dx}{(a+x)^m} = -\frac{1}{m-1}\left[\frac{1}{(a+p)^{m-1}} - \frac{1}{(a+q)^{m-1}}\right].$$

As long as $p > q > -a$ this integration is correct. But if $p > -a > q$ we have

$$\int_q^p \frac{dx}{(a+x)^m} = \int_{-a+\Delta x}^{p} \frac{dx}{(a+x)^m} + \int_q^{-a-\Delta x} \frac{dx}{(a+x)^m}$$

$$= -\frac{1}{m-1}\left[\frac{1}{(a+p)^{m-1}} - \frac{1}{(\Delta x)^{m-1}} + \frac{1}{(-\Delta x)^{m-1}} - \frac{1}{(a+q)^{m-1}}\right].$$

If $m = 2n$ we have

$$\int_q^p \frac{dx}{(a+x)^{2n}} = \frac{2}{2n-1}\cdot\frac{1}{\Delta x^{2n-1}} - \frac{1}{2n-1}\left[\frac{1}{(a+p)^{2n-1}} - \frac{1}{(a+q)^{2n-1}}\right] = \infty,$$

and $$\int_{-\infty}^{\infty} \frac{dx}{(a+x)^{2n}} = \frac{2}{2n-1}\cdot\frac{1}{\Delta x^{2n-1}} = \infty \text{ [the same as } (b_1''')].$$

If $m = 2n+1$ we have

$$\int_q^p \frac{dx}{(a+x)^{2n+1}} = -\frac{1}{2n}\left[\frac{1}{(a+p)^{2n}} - \frac{1}{(a+q)^{2n}}\right], \quad \text{and}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(a+x)^{2n+1}} = 0 \text{ [the same as } (b_2''')].$$

Finally if $m = 1$

$$\int_q^p \frac{dx}{a+x} = \left[\frac{(a+p)^{1-m} - (\Delta x)^{1-m} + (-\Delta x)^{1-m} - (a+q)^{1-m}}{1-m}\right]_{m=1}$$

$$= \log\left(\frac{a+p}{\Delta x}\cdot\frac{-\Delta x}{a+q}\right) = \log\left(\frac{a+p}{-(a+q)}\right); \text{ consquently}$$

$$\int_{-\infty}^{\infty} \frac{dx}{a+x} = \log\left(\frac{a+\infty}{-a+\infty}\right) = \log 1 = 0 \text{ [the same as } (b_3''')].$$

As stated above, Bierens de Haan omitted the infinite term in formulæ ($10_1'$), ($10_2'$). We should then have found according to his formula

$$\int_{-\infty}^{\infty} \frac{dx}{(a+x)^{2n}} = 0,$$

which is absurd for the reason that this integral represents a series of essentially positive terms, the sum of which cannot be zero.

(*To be continued.*)